Higher-Order Finite Element Characterization

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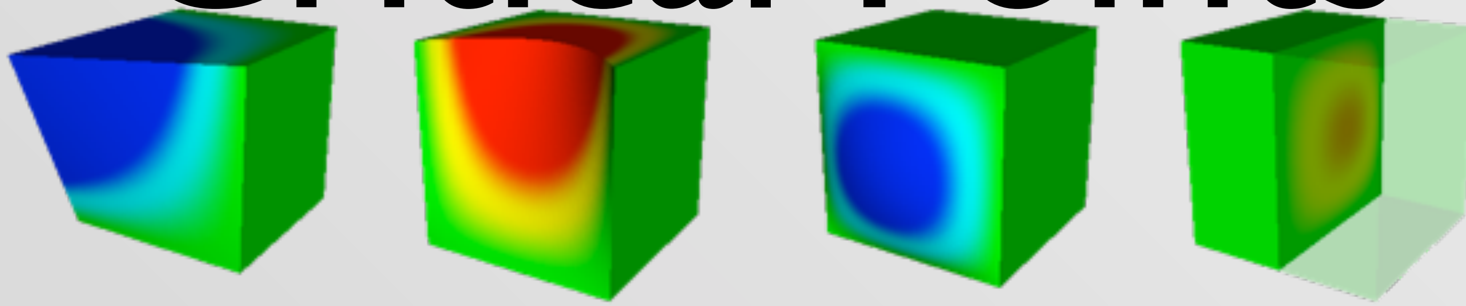
Critical Points

- Linear elements have all extrema at vertices
- Higher order elements do not
- Simple queries become difficult:
  Range, Isocontours, ...
- New types of analysis needed:
  Show redundant degrees of freedom
- Silver lining:
  No need to approximate normals

Critical Points



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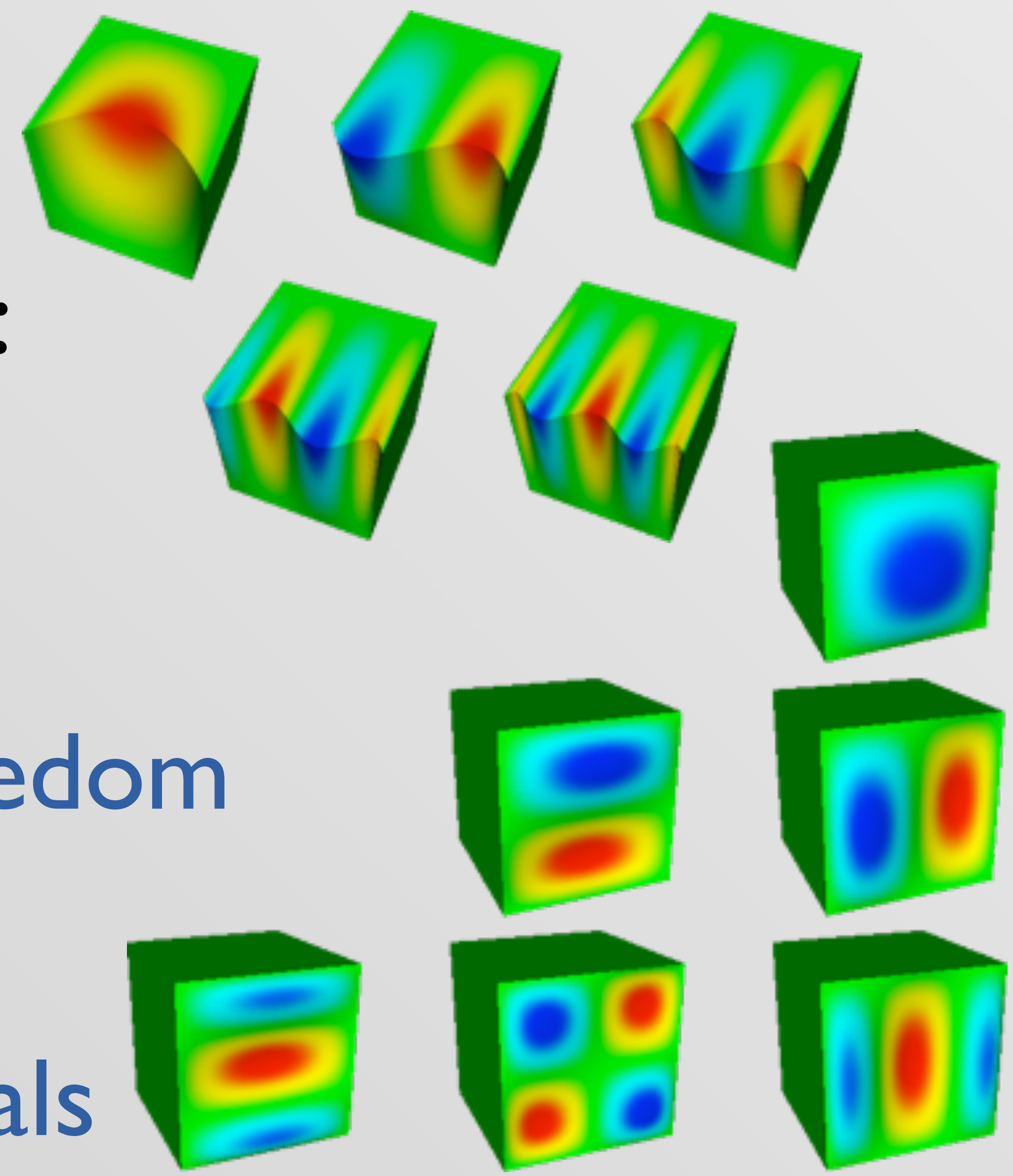
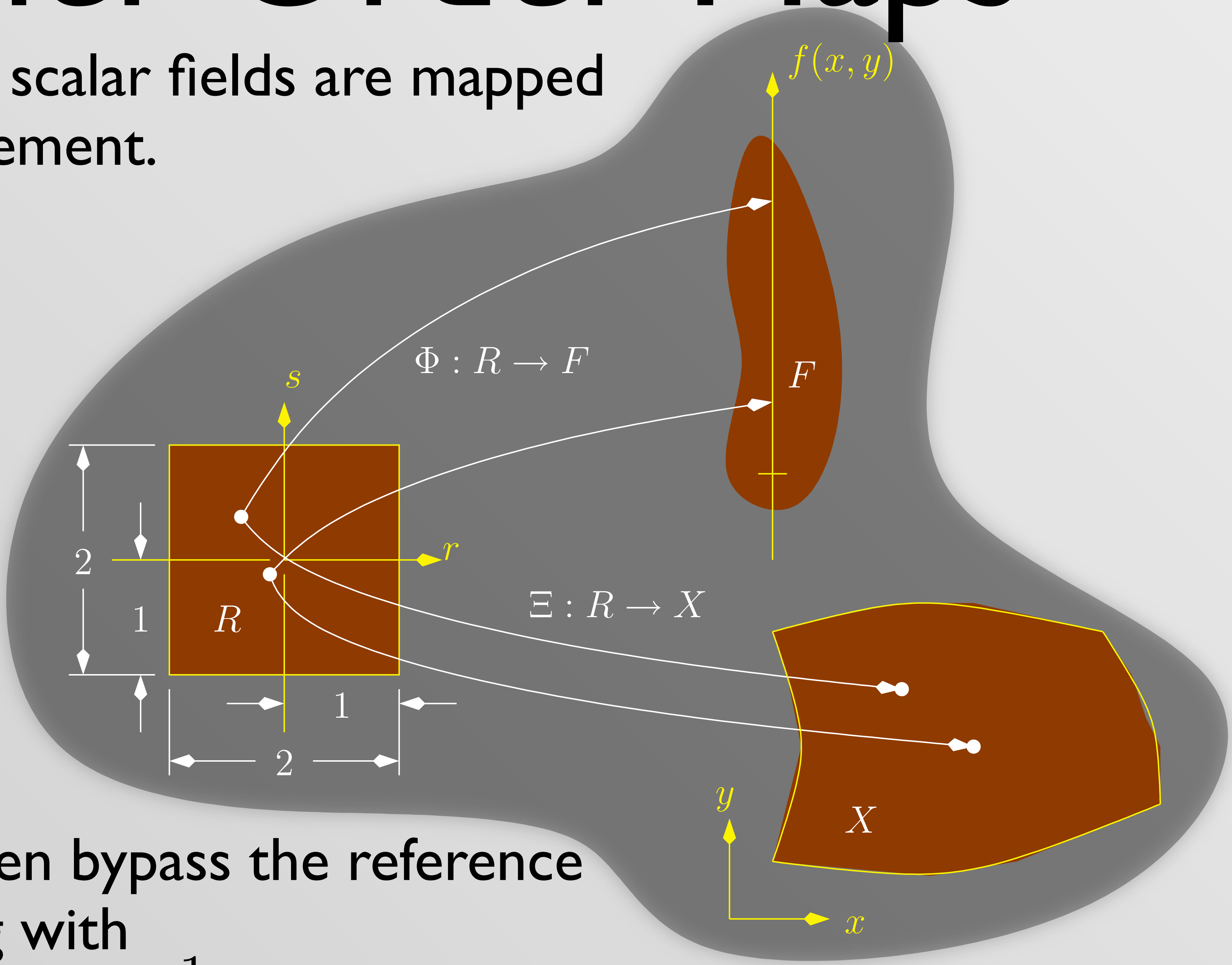


Higher Order Maps

Both geometry and scalar fields are mapped from a reference element.



Linear elements often bypass the reference element by working with

$$f(x, y) = \Phi \circ \Xi^{-1}(x, y) = 0$$

directly, since the inverse of the geometric interpolant is simple.

Isocontouring

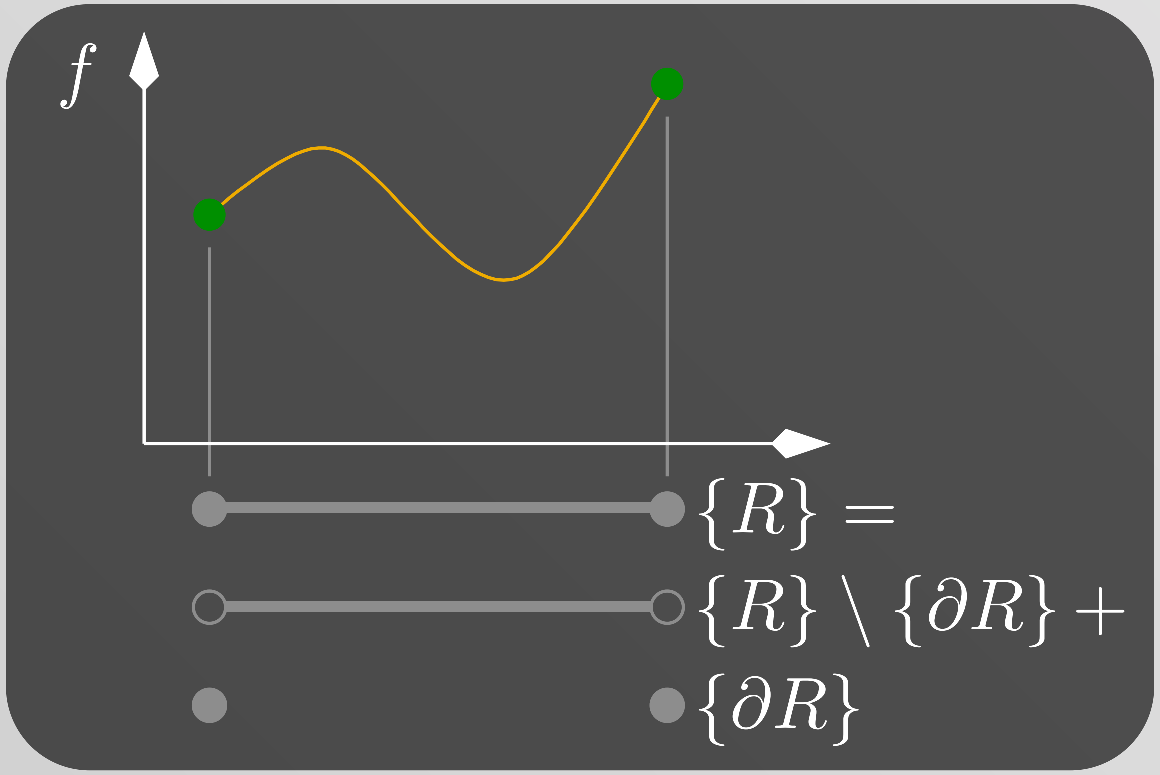
- Where linear isocontouring enumerates
$$f(x, y, z) = 0$$
Higher order isocontouring must solve
$$\Phi \circ \Xi^{-1}(x, y, z) = 0$$
But if Ξ^{-1} is well-behaved, this is
$$\Phi(r, s, t) = 0$$
- Isocontouring requires knowledge of critical points because
  - they indicate non-empty level sets.
  - they generate surface topology.

Critical Points

- So how can we locate extrema?

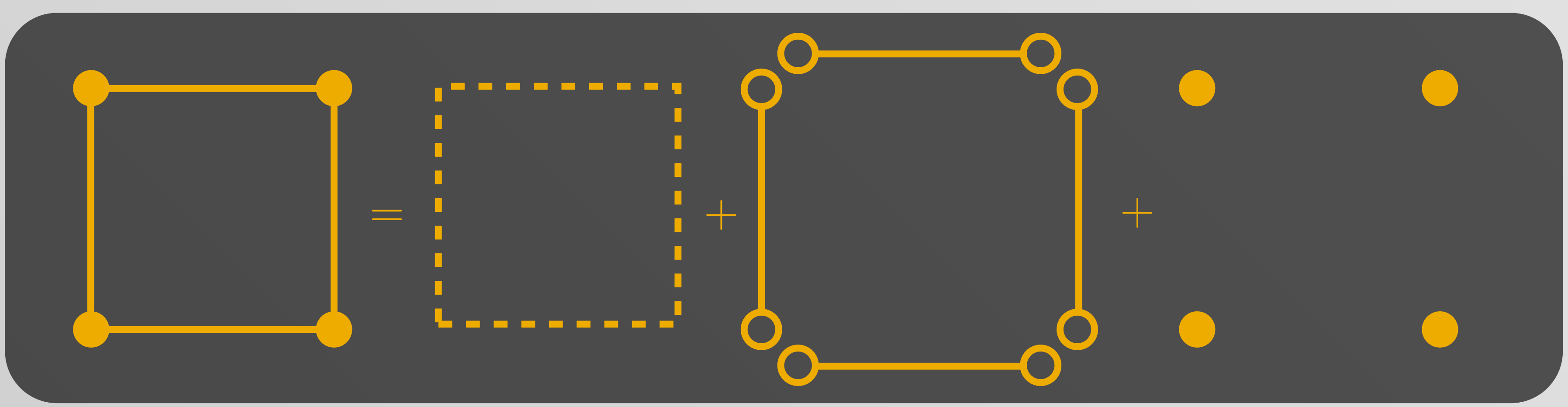
$$\nabla\Phi(r, s, t) = 0$$

- Unfortunately, it's not that easy.
- For closed domains, we must search the open domain and then its closure;



Critical Points

- For closed domains, we must search the open domain and then its closure;
  - The closure may itself need to be decomposed into an open domain and *its* closure;
  - And the same for the new closure!



Restricted Domains

- For hexahedra, restricting the function to a face or edge is simple substitution.

- In general, for edges:

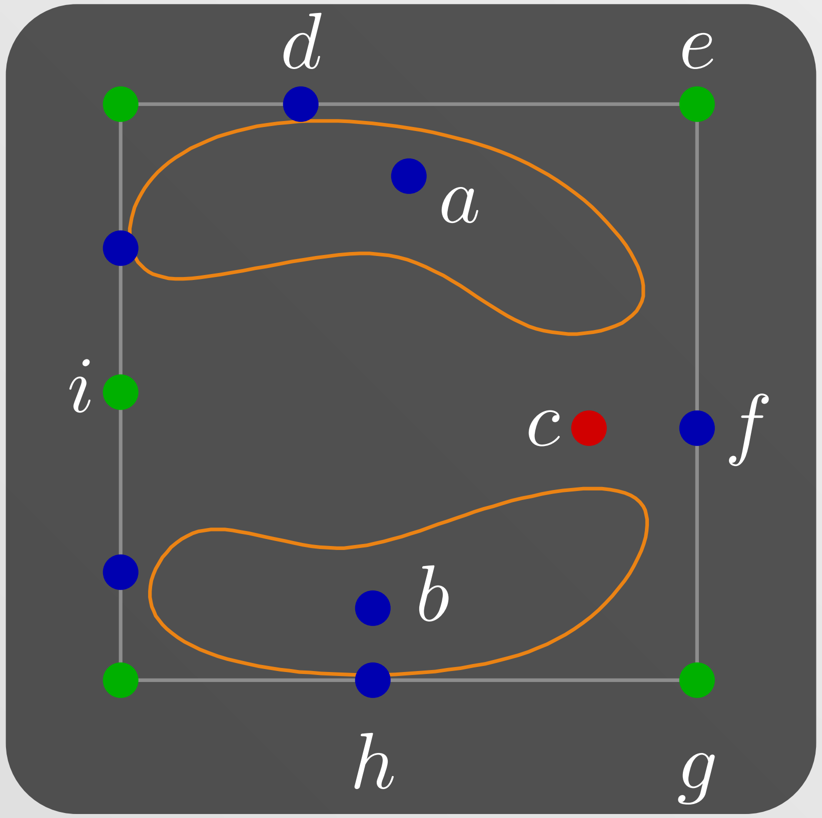
$$\left(\frac{\partial\Phi}{\partial r} + a\frac{\partial\Phi}{\partial s} + b\frac{\partial\Phi}{\partial t}\right)(r, s(r), t(r)) = 0$$

and for faces:

$$\left(\frac{\partial\Phi}{\partial r} + a\frac{\partial\Phi}{\partial t}, \frac{\partial\Phi}{\partial r} + b\frac{\partial\Phi}{\partial t}\right)(r, s, t(r, s) = 0$$

- Note that zeros can occur even where $\nabla\Phi \neq 0$

Critical Points Plus ...



- Critical points contain useful information about isosurfaces, but not enough to march across them.
- Marching requires a decomposition of the volume that meets sampling criteria:
  - Multiple edge intersections: bad!
  - Face-only intersections: bad!
  - Interior components: bad!

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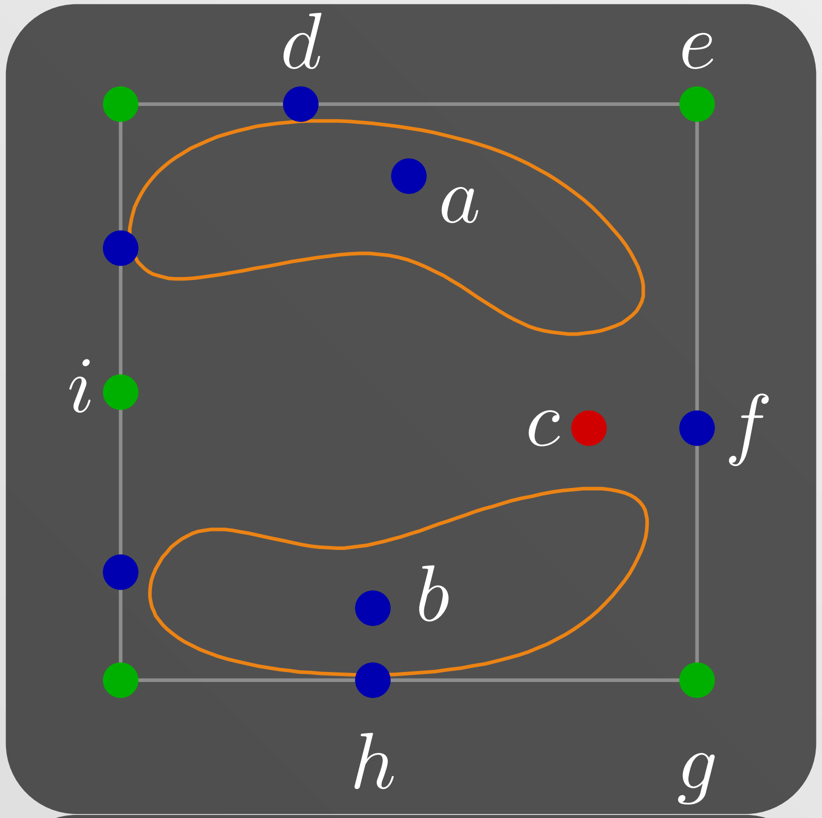

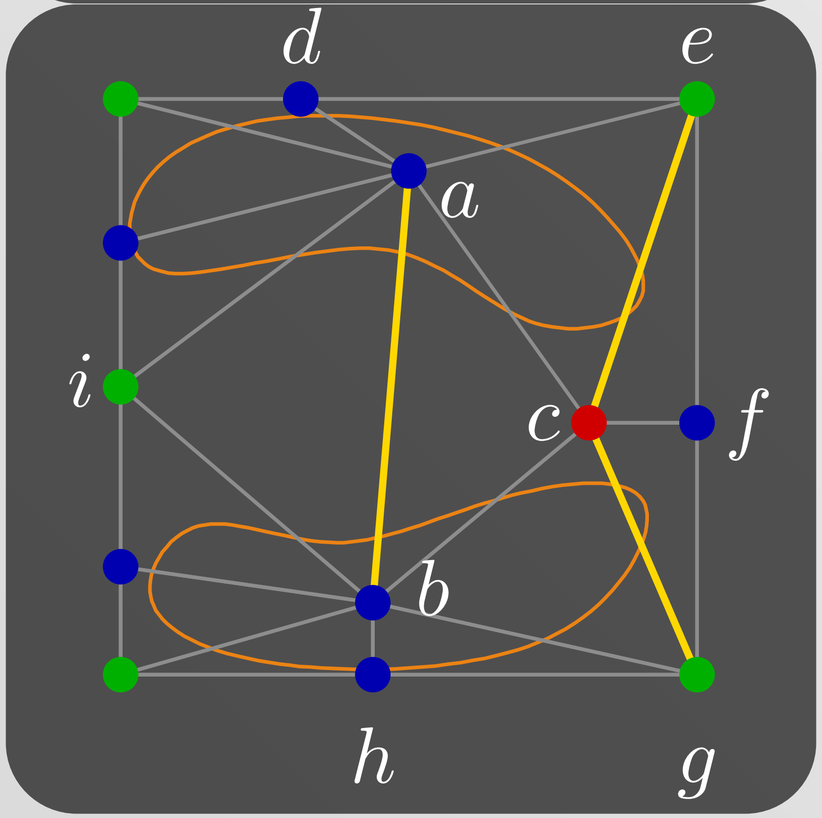

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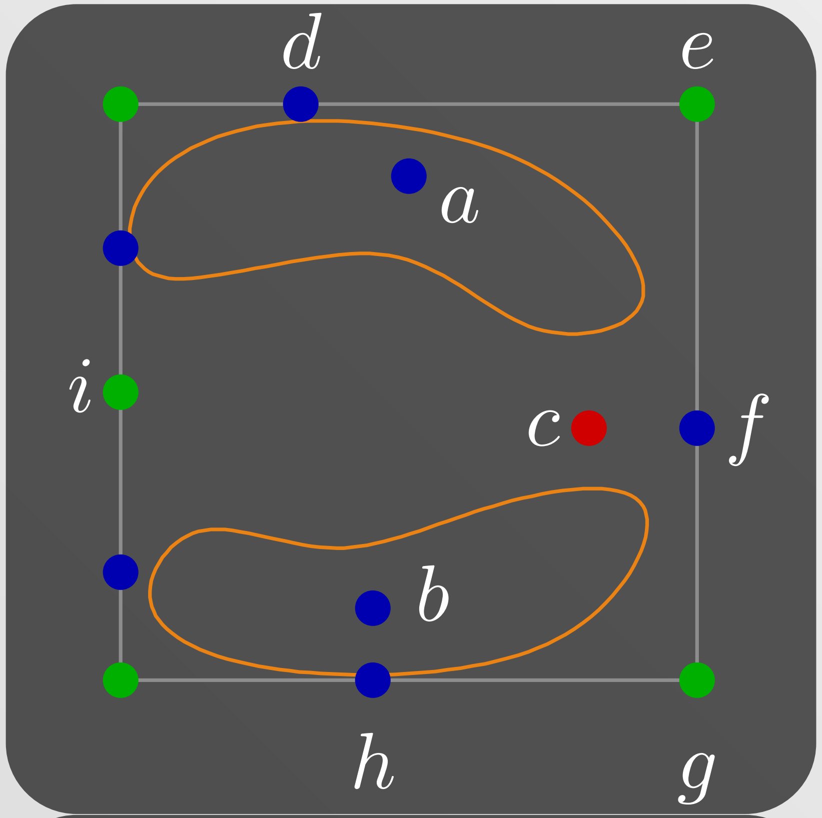

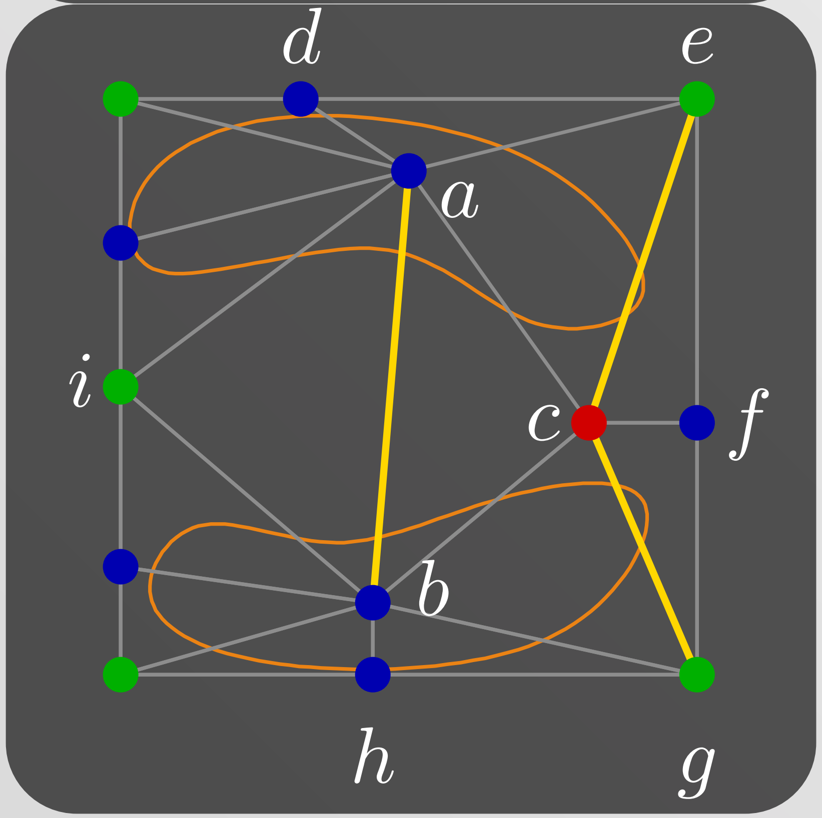

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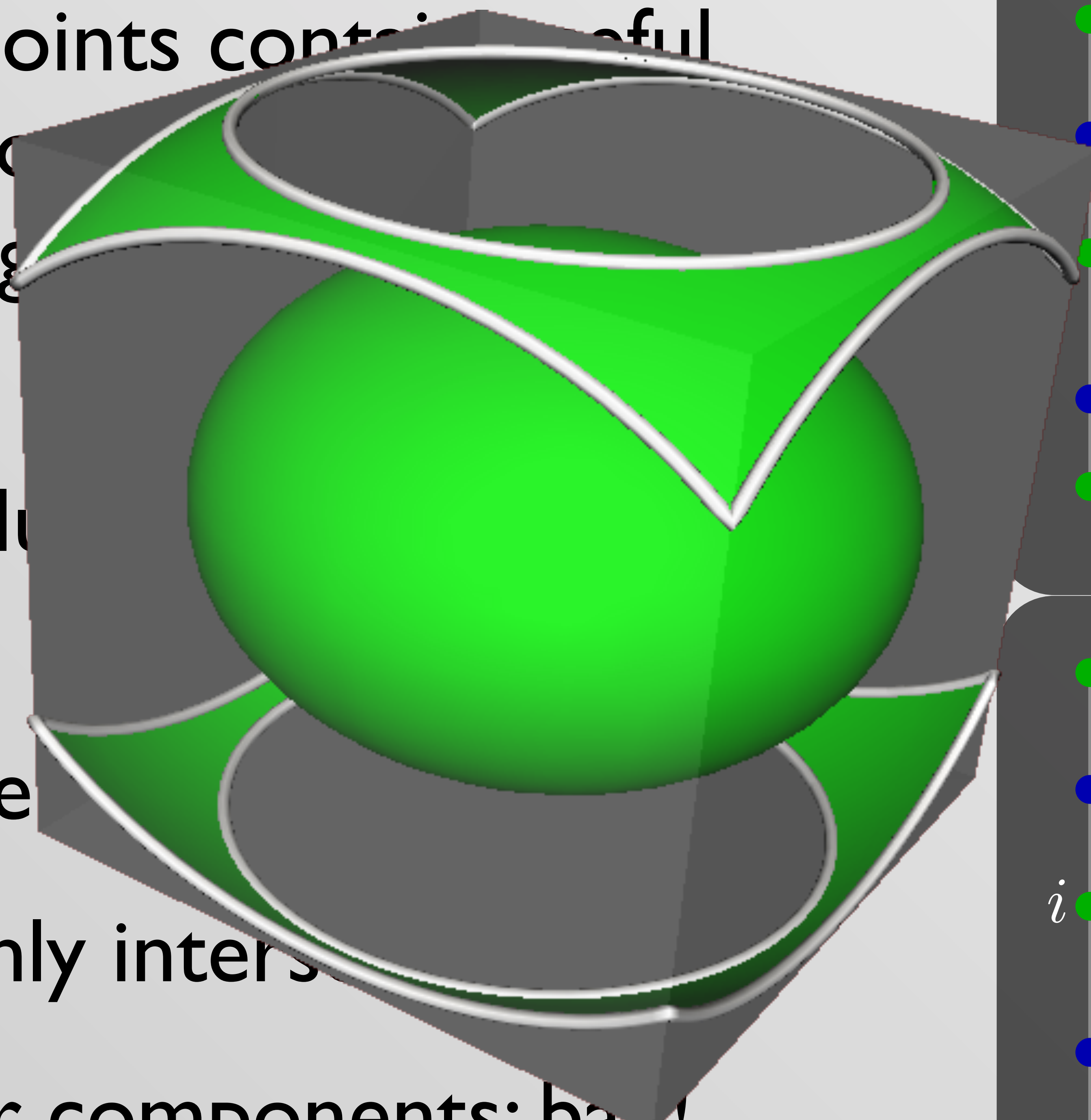
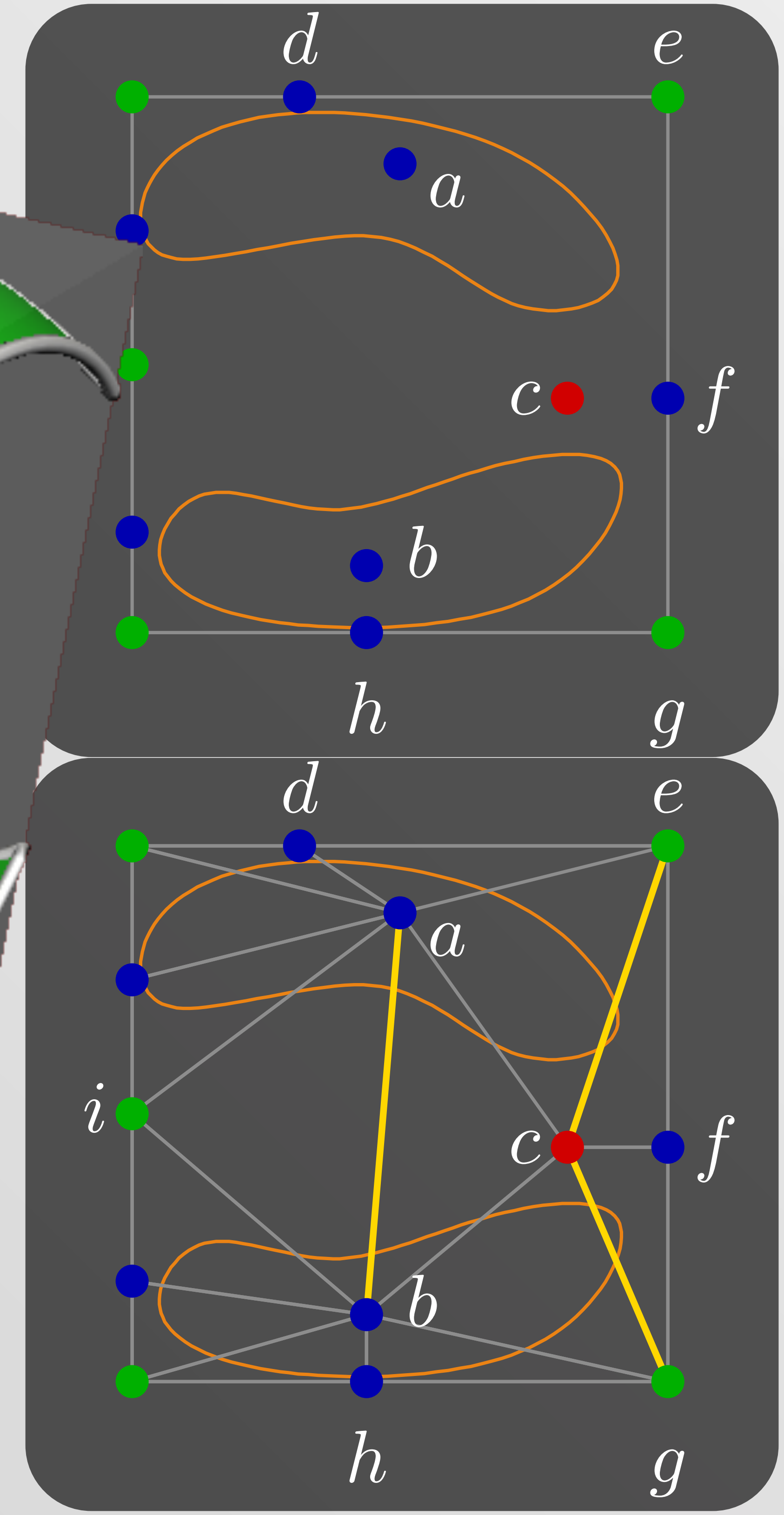
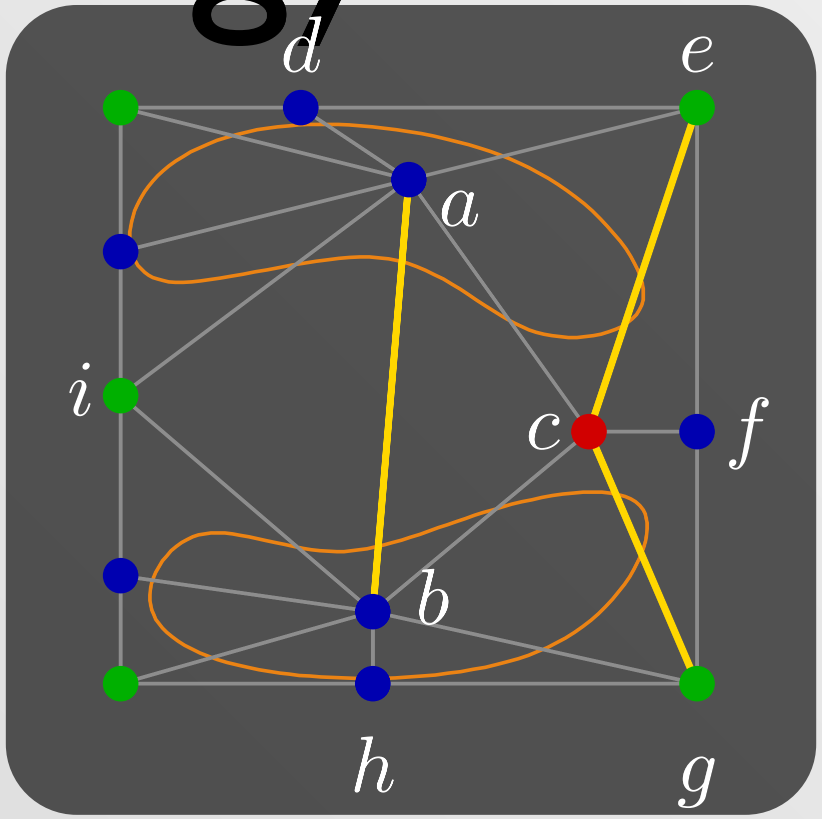




Correcting Topology



- Possible solutions
  - Edge swaps; may not work in 3D
  - Search edges (& faces) for new critical points; generates larger partitions
- Limitations
  - Even when conditions aren't met, it will work – it just won't terminate.

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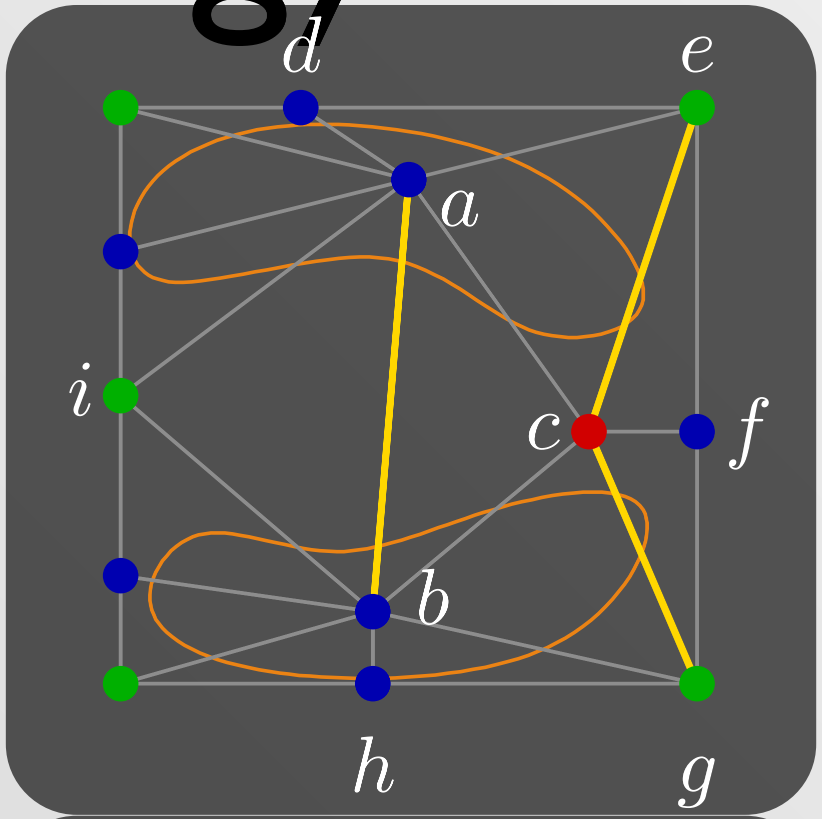
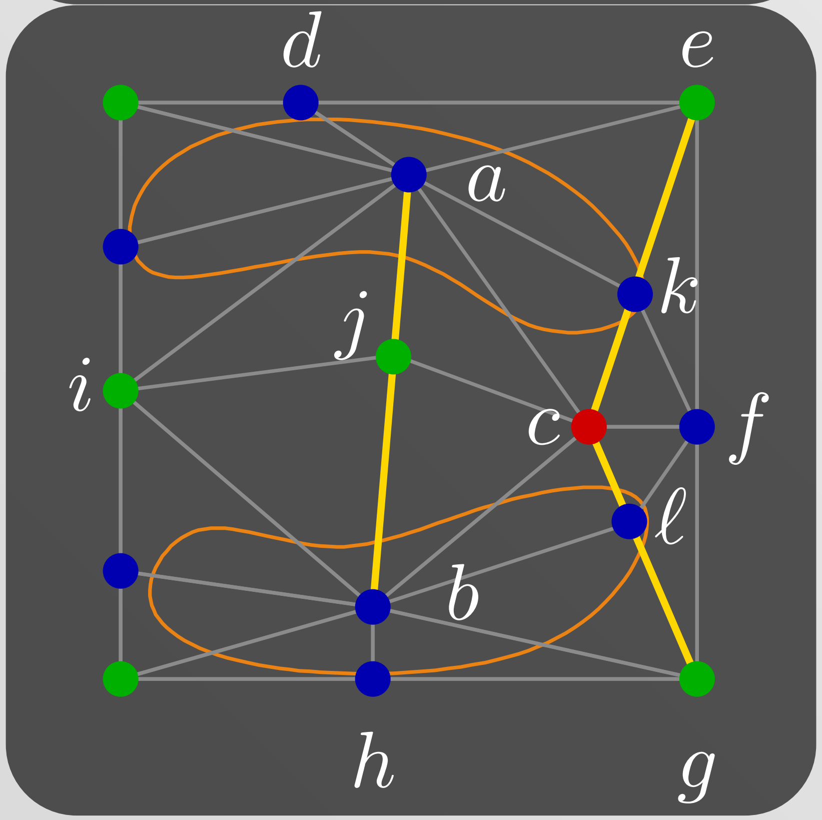
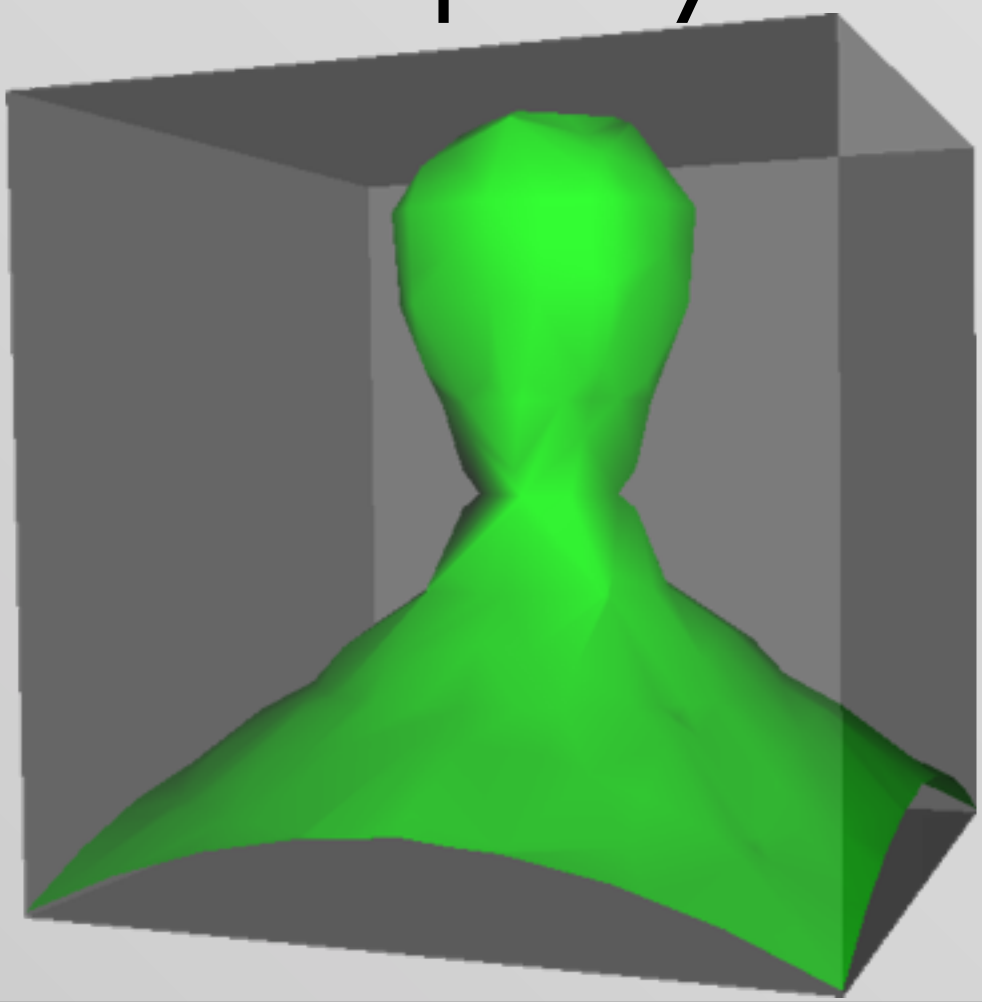
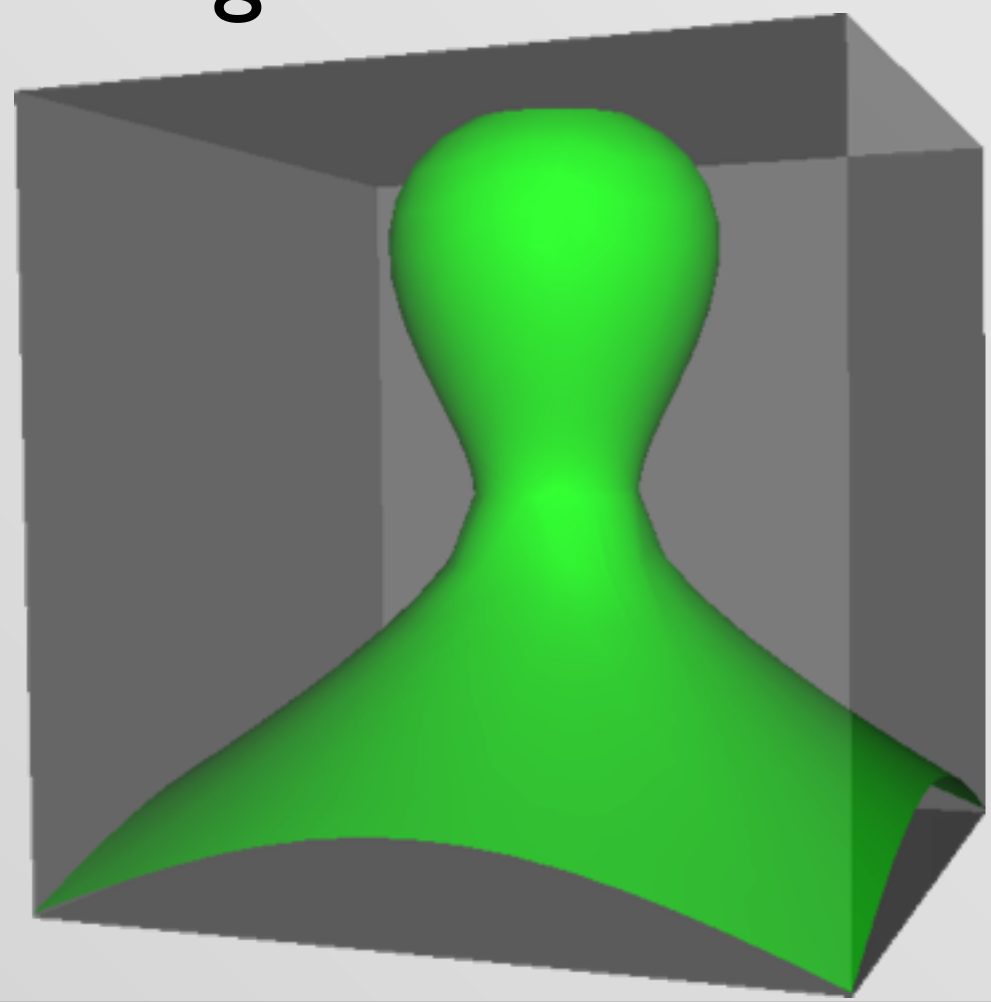




Partition For Analysis

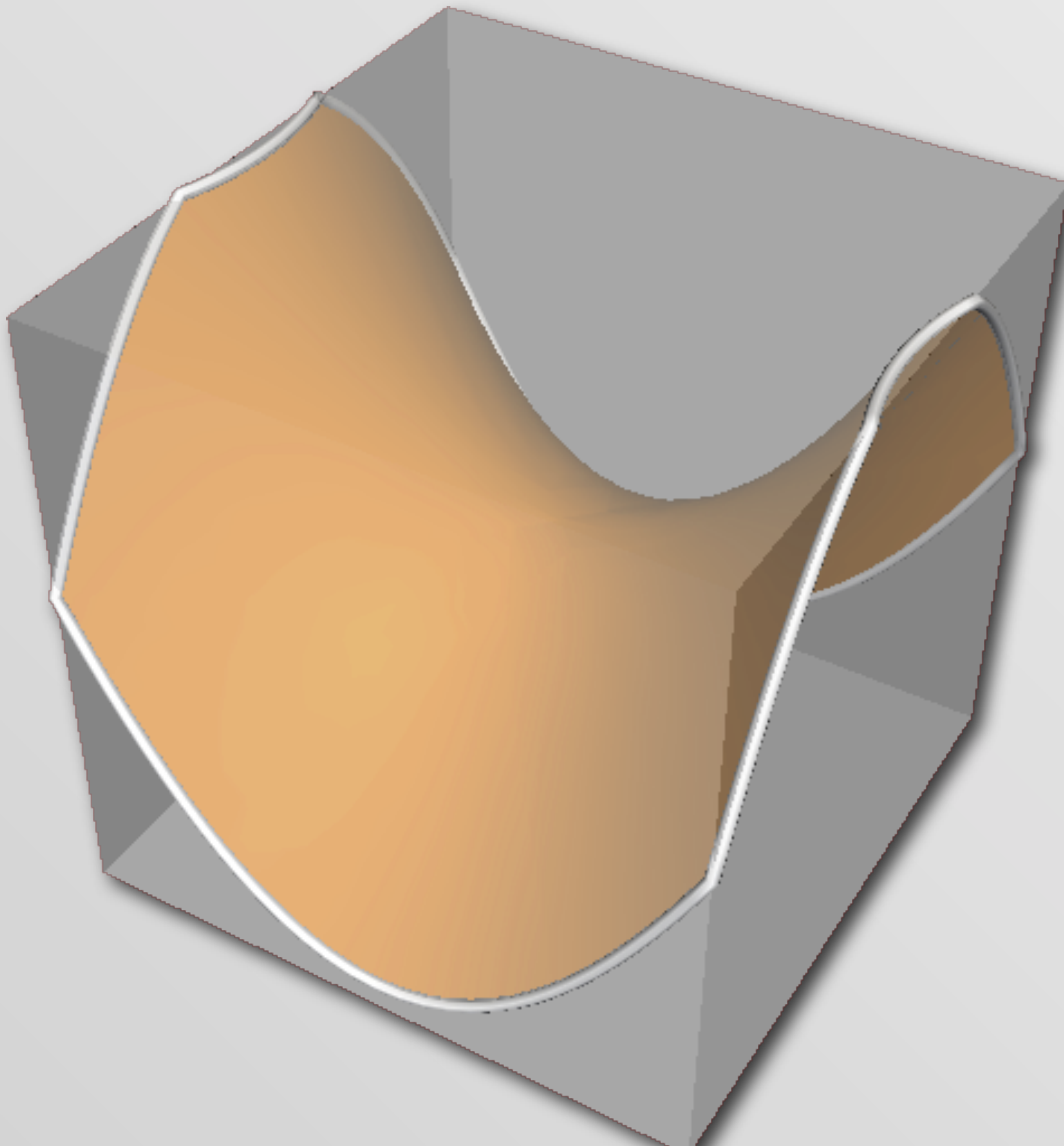
- ... not for linearization!
- Apply Marching Tetrahedra to the partition, but use the higher order interpolant.
- Adaptively subdivide triangles on isosurface.





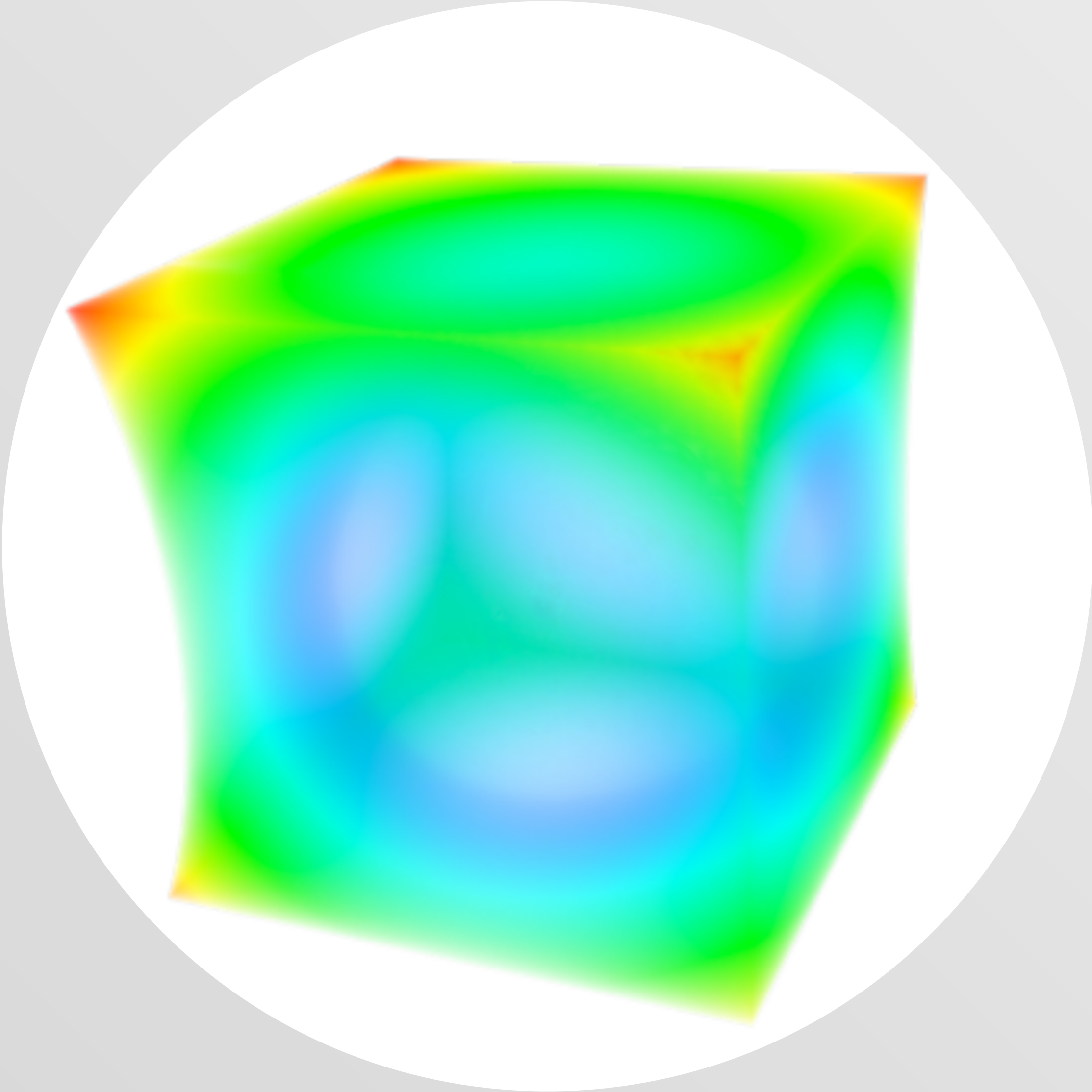
Future Work

- Better polynomial solvers
  - Resultant methods have problems with repeated roots (that touch zero, but do not cross)
- Stronger results for non-isolated critical points
  - Can we tell the first iteration where we can stop and have a valid topology?



To Linearize Or... Not

- When no higher-order equivalent exists, it may be best to approximate...



- But at least we can guarantee the true range of scalar values is represented.